Stability of the vortex lattice in D-wave superconductors

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Abstract

Use is made of Onsager's hydrodynamic equation to derive the vibration spectrum of the vortex lattice in d-wave superconductor. In particular the rhombic lattice (*i.e.* the 45° tilted square lattice) is found to be stable for $B > H_{cr}(t)$. Here $H_{cr}(t)$ denotes the critical field at which the vortex lattice transition takes place.

Since the discovery of the triangular vortex lattice in type II superconductors by Abrikosov [1] and others [2,3], the vibrational modes of the vortex lattice have been studied by a number of people [4]. In these works the stability of the triangular vortex lattice in an s-wave superconductor is established.

The discovery of hole-doped high T_c cuprate superconductivity by Bednorz and Müller [5] in 1986 and the recent realization [6] that d-wave superconductivity is involved may give a new twist on the whole subject.

We have shown earlier that the rhombic vortex lattice (or the 45° tilted square lattice) is stable in the vicinity of the upper critical field in a d-wave superconductor in a magnetic field parallel to the c axis [7]. More recently we have shown [8] the vortex lattice transition from the triangular lattice to the square lattice takes place at a small magnetic field $H_{cr}(t) \sim \kappa^{-1} H_{c2}(t)$, which implies that the vortex lattice should be rhombic in the overwhelming region in the B-T phase diagram (see Fig. 1). Here κ is the Ginzburg-Landau parameter and we have H_{cr} about a few Tesla in YBCO and Bi2212 at low temperatures. Indeed this critical field $H_{cr}(t)$ is consistent with the observation of rhombic vortex lattice by SANS [9] and by STM imaging [10] in monocrystals of YBCO at 3 Tesla.

The object of this paper is to study the vibrational spectrum of the 45° tilted square vortex lattice in a magnetic field. First following Fetter et al. [4], we study Onsager's and Landau's hydrodynamic equation [11,12] for a vortex lattice. Basically we assume that the vortex moves with the local velocity generated by other vortices. We find that the square vortex lattice is unstable when $B < H_{cr}(t)$ but becomes stable for $B \geq H_{cr}(t)$. The vibrational spectrum are determined in the whole Brillouin zone. Second we analyze the vibration spectrum within the time dependent Ginzburg-Landau equation [14] with the Aranov-Hikami-Larkin term [15]. In this limit we have now a set of damped oscillation modes rather than oscillation modes. However the stability condition of the square vortex lattice is the same as in the analysis using Landau's hydrodynamic equation as expected. For example this new vibration spectrum will be crucial in determining the melting transition line where the vortex lattice melts into a vortex liquid.

I. Vibration modes in the square lattice (hydrodynamic limit)

As is well known the Landau's and Onsager's hydrodynamic equation applies when motion of vortices involves no energy dissipation. Unfortunately this condition is never realized for vortices in a d-wave superconductor since there are the low energy extended states attached to every vortex and they certainly dissipate energy whenever the vortex is in motion [16]. Nevertheless it is of great interest to study this idealized limit.

The extended Ginzburg-Landau free energy for the vortex state with $\xi \ll d \ll \lambda$ in d-wave superconductors reduces to [8]

$$\Omega = \frac{2\pi n_\phi \xi^2}{\kappa^2} \sum_L{}' \phi(\mathbf{r}_L - \mathbf{r}_0), \tag{1}$$

$$\phi(\mathbf{r}) = -\epsilon a_2 \xi^2 \kappa^2 \frac{\cos 4\theta_{\mathbf{r}}}{|\mathbf{r}|^4} + K_0\left(\frac{r}{\lambda}\right), \tag{2}$$

where $\epsilon = 31\zeta(5)(-\ln t)/196\zeta(3)^2 \sim 0.114(-\ln t)$, $t = T_c/T$, $a_2 = \frac{20}{3}(\ln 2 + \frac{1}{8}) = 5.454\cdots$[1]; and ξ, d, and λ are the coherence length, intervortex distance and the magnetic penetration depth respectively. Here $\phi(\mathbf{r})$ describes the interaction energy between two vortices; the first term is the core interaction (new to d-wave superconductors) while the second term is the usual magnetic interaction. Following Onsager [12] the equation for the L-th vortex is

$$\dot{\mathbf{u}}_L = -\frac{\bar{\kappa}}{2\pi}(\hat{\mathbf{z}} \times \nabla) \sum_{L' \neq L} (\mathbf{u}_{LL'} \cdot \nabla)\, \phi(\mathbf{r}_{L'} - \mathbf{r}_L), \tag{3}$$

where $\bar{\kappa} = h/2m = \pi/m$ (in quantum unit), $\mathbf{r}_L$ is the position of the L-the vortex, $\mathbf{u}_L$ is the small displacement from the equilibrium position $\mathbf{r}_L^0$ (namely $\mathbf{r}_L = \mathbf{u}_L + \mathbf{r}_L^0$), ∇ denotes the differentiation with respect to $\mathbf{r}_L$. We have set $\mathbf{u}_{LL'} = \mathbf{u}_L - \mathbf{u}_{L'}$, $\mathbf{r}_{LL'} = \mathbf{r}_L - \mathbf{r}_{L'}$. In deriving Eq.(3) the Hamiltonian $H(\mathbf{r}_L)$ in Onsager's theory is replaced by the free energy $\Omega(\mathbf{r}_L)$ obtained in Ref. [8]. In Ref. [8], the square vortex lattice tilted 45° was studied. In the following analysis, however, it is more convenient to use the coordinates system rotated by $\pi/4$ around the z axis, so that vortex lattice axes coincide with the x- and y-axes. This change makes the sign in front of the core interaction term opposite from the one in Ref. [8]. The equilibrium positions of the square vortex lattice become $\mathbf{r}_L^0 = d(m,n)$ $(m, n \in \mathbf{Z})$, where $d = \sqrt{\phi_0/B}$ (ϕ_0 is the flux quantum).

Now introducing a plane wave representation as $\mathbf{u}_L = \mathbf{s}\exp(i(\mathbf{q} \cdot \mathbf{r}_L^0 - \omega t))$, Eq.(3) is rewritten as

$$\begin{aligned} -i\omega s_x &= \alpha s_x + \beta s_y, \\ -i\omega s_y &= \gamma s_x - \alpha s_y, \end{aligned} \tag{4}$$

where

$$\alpha = \frac{\bar{\kappa}}{2\pi\lambda^2} {\sum_L}' (1 - e^{i\mathbf{q}\cdot\mathbf{r}_L}) \left[\frac{x_L y_L}{r_L^2} K_2\left(\frac{r_L}{\lambda}\right) - 20\epsilon a_2 \kappa^2 \xi^4 \lambda^2 \frac{\sin 6\theta_L}{r_L^6} \right], \tag{5}$$

$$\xi = \frac{\beta + \gamma}{2} = \frac{\bar{\kappa}}{4\pi\lambda^2} {\sum_L}' (1 - e^{i\mathbf{q}\cdot\mathbf{r}_L}) \left[\frac{y_L^2 - x_L^2}{r_L^2} K_2\left(\frac{r_L}{\lambda}\right) + 40\epsilon a_2 \kappa^2 \xi^4 \lambda^2 \frac{\cos 6\theta_L}{r_L^6} \right], \tag{6}$$

$$\eta = \frac{\beta - \gamma}{2} = \frac{\bar{\kappa}}{4\pi\lambda^2} {\sum_L}' (1 - e^{i\mathbf{q}\cdot\mathbf{r}_L}) K_0\left(\frac{r_L}{\lambda}\right), \tag{7}$$

where $K_0(x)$ and $K_2(x)$ are the modified Bessel functions. The vibration frequency ω is then given by

$$\omega^2 = -\alpha^2 - \beta\gamma = \eta^2 - \alpha^2 - \xi^2. \tag{8}$$

Let us consider the long wave length limit ($q = |\mathbf{q}| \ll 1$) of the vibration spectrum. Then we obtain

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[1] Note that a_2 is related to the constant a_1 given in Ref. [8] as $a_2 = 5a_1/2\pi$.

$$-\xi + i\alpha = \frac{\bar{\kappa}}{32\pi\lambda^2} q^2 d^2 \left[(\Sigma_{24} - \frac{a(B)}{\mu^2}\Lambda_8) e^{2i\chi} + (\Sigma_{24} - 8\Sigma_{xy4} - \frac{a(B)}{\mu^2}\Lambda_4) e^{-2i\chi} \right], \tag{9}$$

$$\eta = \frac{\bar{\kappa}}{16\pi\lambda^2} q^2 d^2 \Sigma_1, \tag{10}$$

where χ is the angle $\mathbf{q}$ makes with the x axis, we have set

$$a(B) = \frac{40\epsilon a_2 \kappa^2 \xi^4}{\phi_0^2} B^2 \simeq 0.145025 \left(\frac{B}{H_{cr}} \right)^2, \tag{11}$$

where $\mu = d/\lambda (\ll 1)$ and

$$H_{cr}(t) \equiv \phi_0 \sqrt{\frac{4\pi C - 1}{10\epsilon a_2 \kappa^2 \xi^4 (\Lambda_8 + \Lambda_4)}} \simeq 0.48(-\ln t)^{-1/2} \frac{H_{c2}(t)}{\kappa}. \tag{12}$$

We have also used the notations

$$\Sigma_1 = d^{-2} \sum_L{}' r_L^2 K_0(r_L/\lambda) = \frac{8\pi}{\mu^4} + O(1), \tag{13}$$

$$\Sigma_{24} = d^{-2} \sum_L{}' r_L^2 K_2(r_L/\lambda) = \frac{16\pi}{\mu^4} - \frac{2}{\mu^2} + O(1), \tag{14}$$

$$\Sigma_{xy4} = d^{-2} \sum_L{}' \frac{x_L^2 y_L^2}{r_L^2} K_2(r_L/\lambda) = \frac{2\pi}{\mu^4} - C\frac{2\pi}{\mu^2} + O(1), \tag{15}$$

$$\Lambda_8 = d^4 \sum_L{}' \frac{1}{r_L^4} \cos(8\theta_L) = 5.0306\cdots, \tag{16}$$

$$\Lambda_4 = d^4 \sum_L{}' \frac{1}{r_L^4} \cos(4\theta_L) = 3.1512\cdots, \tag{17}$$

where $C = 0.10331$. The quantities Σ_1, Σ_{24}, Σ_{xy4} and C are quoted from [4] for the sake of the readers convenience. Using these, we have

$$\omega^2 \left(\frac{32\pi}{\bar{\kappa} q^2 \mu^2} \right)^2 = = \frac{32\pi}{\mu^6} \left[(2 + a(B)\Lambda_8) - (16\pi C - 2 - a(B)\Lambda_4)\cos 4\chi \right] + O(\mu^{-4}). \tag{18}$$

Finally, we obtain

$$\omega^2 \simeq \left(\frac{eB}{mc} \right)^2 q^4 \lambda^2 d^2 \left[\left(0.01989 + 0.007257 \left(\frac{B}{H_{cr}} \right)^2 \right) - \left(0.03176 - 0.004546 \left(\frac{B}{H_{cr}} \right)^2 \right) \cos 4\chi \right]. \tag{19}$$

The condition to have real spectrum ($\omega^2 > 0$) is $B \geq H_{cr}$. We see immediately that the square vortex lattice is unstable for $B < H_{cr}$ for some χ, while it becomes stable for all χ for $B > H_{cr}$ as it should be. Exactly at the transition point $B = H_{cr}$ $\omega^2 = 0$ at $\chi = 0, \pi/2, \pi$ and $3\pi/2$.

In the previous paper [8], we derived the critical field as $H_{cr}(t) \simeq 0.52(-\ln t)^{-1/2} H_{c2}(t)/\kappa$ by minimizing the free energy of the extended Landau Ginzburg theory. This and the result obtained here in Eq. (12) appear to be fully consistent.

II. Thermal fluctuation of the vortex lattice

Let us consider thermal fluctuation of the square vortex lattice ($B > H_{cr}$) based on the Landau-Ginzburg free energy (1). This provides us with another way to study the quantities α, β and γ introduced in the last section.

The small deviation of the free energy $\Delta\Omega$ due to the lattice vibration is

$$\Delta\Omega = \frac{\pi n_\phi \xi^2}{\kappa^2} \sum_L{}' \sum_{\mu,\nu=x,y} (u_{L\mu} - u_{0\mu}) \phi_{\mu\nu}(\mathbf{r}_L^0)(u_{L\nu} - u_{0\nu}), \tag{20}$$

where $u_{L\mu}$ denotes the μ-component of the small deviation vector $\mathbf{u}_L$ and $\phi_{\mu\nu}(\mathbf{r}) = \partial_\mu \partial_\nu \phi(\mathbf{r})$. Introducing the Fourier transformation $u_{L\mu} = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}^0} s_\mu(\mathbf{q})$ (the reality condition $s_\mu(-\mathbf{q}) = s_\mu(\mathbf{q})^*$ is imposed), we have

$$\begin{aligned}
\Delta\Omega &= \frac{2\pi n_\phi \xi^2}{\kappa^2} \sum_L{}' \sum_{\mu,\nu,\mathbf{q}} s_\mu(\mathbf{q})^* s_\nu(\mathbf{q})(1 - e^{i\mathbf{q}\cdot\mathbf{r}^0})\phi_{\mu\nu}(\mathbf{r}_L^0) \\
&= \frac{4\pi^2 n_\phi \xi^2}{\kappa^2 \bar{\kappa}} \sum_{\mathbf{q}} \Big(s_x(\mathbf{q})^*, s_y(\mathbf{q})^*\Big) \begin{pmatrix} -\gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} s_x(\mathbf{q}) \\ s_y(\mathbf{q}) \end{pmatrix} \\
&= \frac{4\pi^2 n_\phi \xi^2}{\kappa^2 \bar{\kappa}} \sum_{\mathbf{q}} \left(\omega_+ |s_+(\mathbf{q})|^2 + \omega_- |s_-(\mathbf{q})|^2\right),
\end{aligned} \tag{21}$$

where $s_\pm(\mathbf{q})$ denotes a suitable unitary transformation of $s_\mu(\mathbf{q})$ and

$$\begin{aligned}
&\omega_\pm = \eta \pm \sqrt{\alpha^2 + \xi^2} \\
&= \left(\frac{\bar{\kappa}}{32\pi\lambda^2} q d^2\right) \left[\frac{16\pi}{\mu^4} \pm \left(\frac{16\pi}{\mu^4} - \frac{(2 + a(B)\Lambda_8) - (16\pi C - 2 - a(B)\Lambda_4)\cos 4\chi}{\mu^2}\right)\right] && (22) \\
&= \left(\frac{eB}{mc}\right) q^2 \lambda^2 \Bigg[\frac{1 \pm 1}{2} \mp \left(\frac{d}{\lambda}\right)^2 \Bigg(\left(0.01989 + 0.007257 \left(\frac{B}{H_{cr}}\right)^2\right) \\
&\qquad - \left(0.03176 + 0.004546 \left(\frac{B}{H_{cr}}\right)^2\right) \cos 4\chi\Bigg)\Bigg]. && (23)
\end{aligned}$$

The stability condition $\omega_\pm > 0$ leads to $\eta > \sqrt{\alpha^2 + \xi^2}$ or $B > H_{cr}$. Note that this condition is exactly the same as the one in the previous section. Evaluating the path integral $\int \mathcal{D}s \; (|s_x|^2 + |s_y|^2) e^{-\beta\Omega} / \int \mathcal{D}s \; e^{-\beta\Omega}$, we obtain

$$\sum_{\mathbf{q}} \langle |s_x(\mathbf{q})|^2 + |s_y(\mathbf{q})|^2 \rangle = kT \frac{\kappa^2 \bar{\kappa}}{4\pi^2 n_\phi \xi^2} \sum_{\mathbf{q}} \left(\frac{1}{\omega_+(\mathbf{q})} + \frac{1}{\omega_-(\mathbf{q})}\right). \tag{24}$$

Studying this expression we will be able to study the vortex lattice melting from the square vortex lattice. Details on the vortex lattice melting will be reported in a separate paper [13].

III. Time dependent Ginzburg Landau equation approach

In a d-wave superconductor the vortex motion cannot be purely hydrodynamic, since the vortex motion always accompanied with the energy dissipation as seen from a number of experiments involving the flux flow resistance in high T_c cuprate superconductors. Therefore

the vibration models for the vortex lattice should be described better by the time-dependent Ginzburg Landau equation [14], supplemented by the term which accounts for the Hall effect [15] at least qualitatively.

In the vicinity of $T = T_c$ the time dependent Ginzburg Landau (TDGL) equation is the same as that in s-wave superconductors [14]. Neglecting the term which gives rise to the Hall effect we obtain

$$\frac{\pi}{8T_c}\frac{\partial}{\partial t}\Delta(\mathbf{r}) = N_0^{-1}\frac{\partial\Omega}{\partial\Delta^*(\mathbf{r})}, \tag{25}$$

where Ω is the free energy defined in Ref. [8]. Assuming the usual product expression $\Delta(\mathbf{r}) = \Delta\prod_i f(\mathbf{r}-\mathbf{r}_i)$, we obtain

$$\frac{\pi}{8T_c}\Delta^2\mathbf{v}_i\frac{1}{2}\left|\frac{\partial f(\mathbf{r}-\mathbf{r}_i)}{\partial\mathbf{r}_i}\right|^2\prod_{j\neq i}|f(\mathbf{r}-\mathbf{r}_i)|^2 = N_0^{-1}\frac{\partial\Omega}{\partial\mathbf{r}_i}, \tag{26}$$

which reduces to

$$\mathbf{v}_i = \frac{16T_c}{\pi}\frac{\xi^2(T)}{\Delta^2(T)}N_0^{-1}\frac{\partial\Omega}{\partial\mathbf{r}_i}. \tag{27}$$

The right hand side of Eq.(27) is evaluated as in the case of the hydrodynamic limit, and obtain two frequencies

$$\omega_\pm = -iA_0(\eta\pm\sqrt{\xi^2+\alpha^2}), \tag{28}$$

where $\eta\pm\sqrt{\xi^2+\alpha^2}$ is calculated in Eq. (22) and

$$A_0 = \frac{14\zeta(3)}{\pi^3}\frac{E_F}{T_c}. \tag{29}$$

The stability of the vortex lattice requires $\eta > \sqrt{\xi^2+\alpha^2}$, which is satisfied for $B > H_{cr}(t)$.

Finally the gauge invariance of TDGL requires that Eq.(25) has to be replaced by

$$\frac{\pi}{8T_c}(1-i\sigma)\frac{\partial}{\partial t}\Delta(\mathbf{r}) = N_0^{-1}\frac{\partial\Omega}{\partial\Delta^*(\mathbf{r})}, \tag{30}$$

where

$$\sigma = \frac{4T_c}{\pi}\frac{\partial\ln T_c}{\partial\mu}. \tag{31}$$

Then in the presence of the Hall effect Eq.(28) should be replaced by

$$\omega_\pm = \frac{\sigma-i}{1+\sigma^2}A_0(\eta\pm\sqrt{\xi^2+\alpha^2}). \tag{32}$$

Therefore in this general situation the vibration becomes damped oscillation. However, the stability condition is the same as before.

Concluding Remarks
Limiting ourselves to the square vortex lattice, we have studied the oscillation mode of the vortex lattice first by the hydrodynamic equation and then by the time-dependent Ginzburg Landau equation. Both analysis indicates that the square vortex lattice is stable for $B > H_{cr}(t)$, while it becomes unstable for $B < H_{cr}(t)$. Of particular interest is that the fluctuation of the vortex lattice diverges linearly like $(B - H_{cr}(t))^{-1/2}$ as B approaches the transition point. This should have a profound implication on the melting of the vortex lattice for example. The related problems will be discussed in future publications.

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FIGURES

Fig. 1 The B-T phsae diagram.

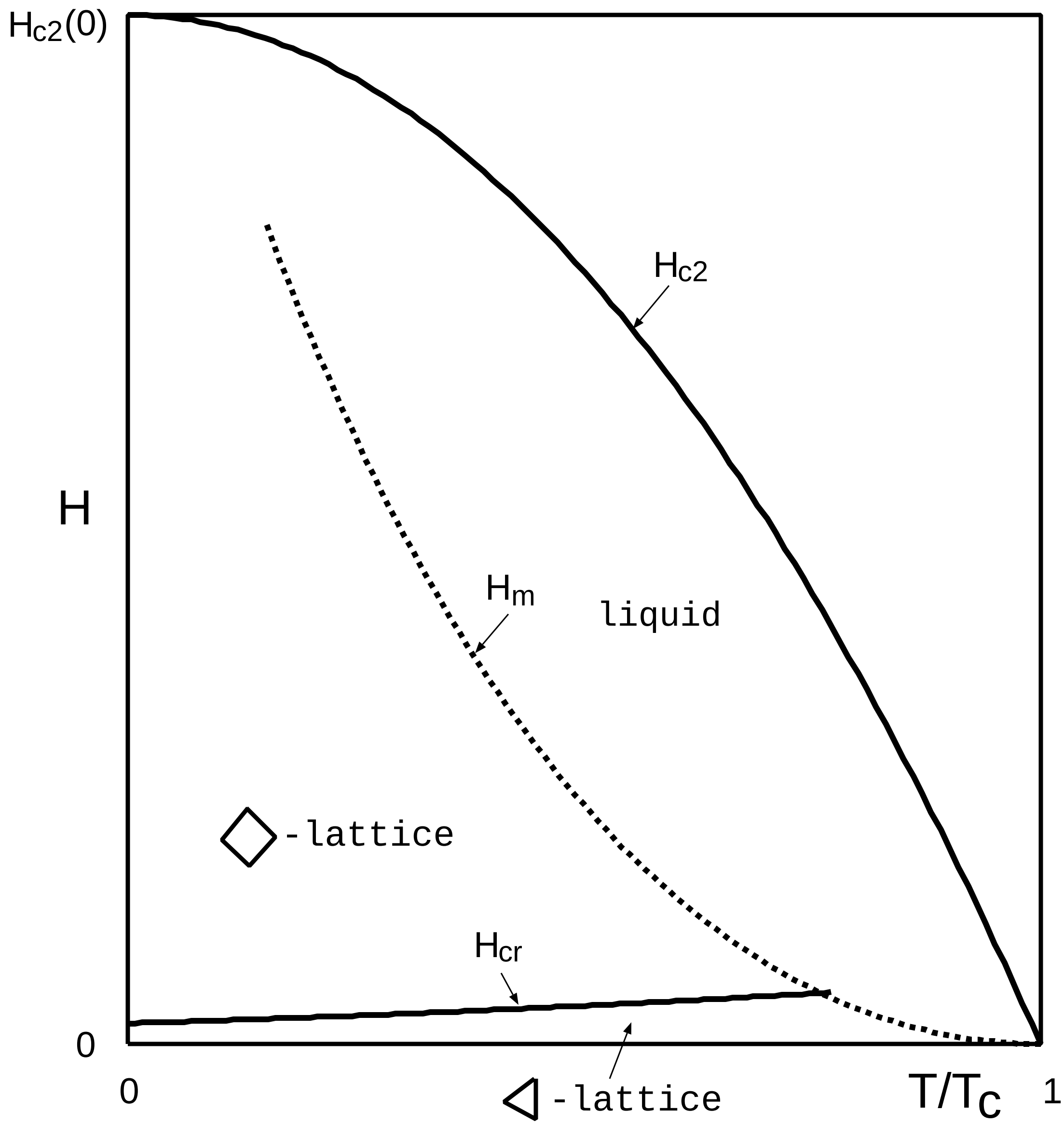

Fig.1